Poses and Motion:

Representations of Motion and Kinematics of Rigid Bodies

“The Heart of Robotics is Motion”, Matt Mason
Representations of Rotations

1) Rotation Matrix (direction cosine matrix)  
$$R = \begin{pmatrix}
\hat{x}_{sb}^x & \hat{y}_{sb}^x & \hat{z}_{sb}^x \\
\hat{x}_{sb}^y & \hat{y}_{sb}^y & \hat{z}_{sb}^y \\
\hat{x}_{sb}^z & \hat{y}_{sb}^z & \hat{z}_{sb}^z \\
\end{pmatrix} \in SO(3)$$

2) Exponential Coordinates (Axis-angle)  
$$\hat{\omega} \theta = \omega \in so(3)$$

3) Euler angles
$$(\alpha, \beta, \gamma) = YPR$$

4) Quaternion
$$q = (q_w, q_x, q_y, q_z)$$
Why so many representations for rotation?

1) Rotation Matrix (direction cosine matrix)
   + Operations on other geometric elements
   + Composition
   - 9 elements for 3 DoF
   - Interpolation

2) Exponential Coordinates (Axis-angle)
   + Minimal representation
   + Intuitive to “visualize”
   - Interpolation
   - Operations on other geometric elements
   - Composition

3) Euler angles
   + Intuitive to “define”
   + Minimal representation
   - Gimbal lock
   - Composition
   - Operations on other geometric elements

4) Quaternion
   + “Almost” minimal representation
   + “Almost” intuitive to “visualize”
   + Interpolation (SLERP)
   - Operations on other geometric elements
Gimbal Lock
Representations of Poses

1) Homogeneous Transformation Matrix

\[
T = \begin{pmatrix}
R & t \\
0 & 1
\end{pmatrix} \in SE(3)
\]

2) Exponential Coordinates (Twist)

\[(v, \omega) \in se(3)\]

Any combination of rotation representation + translation

e.g. \((q, t)\) or \((YPR, t)\)
Why so many representations for pose?

1) Homogeneous Transformation Matrix
   + Operations on other geometric elements
   + Composition
   - 16 elements for 6 DoF

2) Exponential Coordinates (Twist)
   + Minimal representation
   + Good for optimization and iterative error minimization
   - Interpolation
   - Operations on other geometric elements
   - Composition
What is the rotation matrix from S to B?

\[ R_{SB} = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{pmatrix} \]

\[ R_{SB} = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} \]

\[ R_{SB} = \begin{pmatrix}
0 & 0 & -1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} \]
For the same rotation, what are the exponential coordinates?

\[ \omega = (0, \pi/2, 0) \]

\[ \omega = (0, \sqrt{2}/2, \sqrt{2}/2) \cdot \pi/2 \]

\[ \omega = (0, 1, 0) \cdot \pi/2 \]

\[ \omega = (1, 0, 0) \cdot \pi \]
For the same rotation, what is its representation as quaternion?

\[ q = (w = 0, 1, 0, 0) \]

\[ q = (w = 1, 0, 0, 0) \]

\[ q = (w = \sqrt{2}/2, 0, \sqrt{2}/2, 0) \]

\[ q = (w = \sqrt{2}/2, 0, 0, 1) \]
For the same rotation, what are the Euler angles (yaw, pitch, roll)?

\[
\begin{align*}
\text{YPR} &= (0, \pi/2, 0) \\
\text{YPR} &= (0, \pi/2, \pi/2) \\
\text{YPR} &= (0, \sqrt{2}/2, 0)
\end{align*}
\]
What will we learn - Fundamentals of Motion

- Recap of Linear Algebra and Linear Differential Equation
- Representation of rotations
  - Lie Group - Lie Algebra and Exponential Coordinates
- Pose, Homogeneous transformation matrix
- Kinematics of rigid bodies
Recap

- So far: Poses, motion between two time steps t0 and t1
  (we used velocity, e.g. \( \dot{p}(t) = \omega \times p(t) \), only to derive exponential coordinates)

- Now: Continuous change in pose over time -> velocity
Given the orientation $R(t)$ of a rotating frame as a function of time $t$, what is its angular velocity?

\[
\dot{\mathbf{p}}_s(t) = \mathbf{w}_s \times \mathbf{p}_s(t) = \begin{bmatrix} \mathbf{w}_s \end{bmatrix} \mathbf{p}_s(t)
\]

\[
\dot{\mathbf{p}}_s(t) = R_{sb} \mathbf{p}_b \quad \Rightarrow \quad \dot{\mathbf{p}}_s(t) = \dot{R}_{sb} \mathbf{p}_b = \dot{R}_{sb} R_{sb}^T \mathbf{p}_s
\]

\[
\Rightarrow [\mathbf{w}_s] = \dot{R}_{sb} \cdot R_{sb}^T
\]
Rotational Velocity in Space and Body Frame

Def: Rotational velocity in space frame: Given the orientation $R_{sb}(t)$ of a rotating frame $\{b\}$ at time $t$. The (instantaneous) angular velocity vector $w$ of frame $\{b\}$ defined in frame $\{s\}$ is

$$[w_s] = \dot{R}_{sb} R_{sb}^{-1}$$

The same velocity defined in frame $\{b\}$ is

$$[w_b] = R_{sb}^{-1} \dot{R}_{sb}$$
Integrating Angular Velocity into Rotation Matrix

Given the constant angular velocity of a body, what is the orientation after \( t \) seconds?

\[
R_{SB}(t) = e^{[\omega_S]t} \cdot M
\]

assuming \( R_{SB}(0) = M \)
Kinematics of Rigid Bodies - Twists as Rigid Body Velocity

\[ T_{sb}(t) = (R_{sb}(t), \rho(t)) \]

\[ \mathbf{U}_s = \Xi_s = (\mathbf{v}_s, \omega_s), \quad [\omega_b] = R_{sb} R_{sb}^T, \quad \mathbf{v}_s = \dot{\mathbf{p}} + \omega_s \times (-\mathbf{p}) \]

\[ \mathbf{U}_b = \text{Ad}_{R_{sb}} \mathbf{U}_s, \quad [\omega_b] = R_{sb}^T \dot{R}_{sb}, \quad \mathbf{v}_b = R^T \dot{\mathbf{p}} \]
Adjoint Transform

\[ T_{sb} = (R_{sb}, p) \]

\[ A_{dT} = \begin{bmatrix} R_{sb} & 0 \\ \L_P R_{sb} & R_{sb} \end{bmatrix} \]

Transform twist coordinates between different reference frames.
Spatial Twist and Body Twist – Interpretations

\[ \mathbf{U}_s = (v_s, w_s) \]

- \( w_s \): rotation given wrt. \( \{s\} \)
- \( v_s \): velocity of the origin of \( \{b\} \) in \( \{s\} \) coordinates

\[ \mathbf{U}_b = (v_b, w_b) \]

- \( w_b \): rotation given wrt. \( \{b\} \)
- \( v_b \): velocity of origin of \( \{b\} \) wrt. instantaneous \( \{b\} \) frame
Integrating Velocity Twists into Transformation Matrix

$T_{sb}(0) = M$

$T_{sb}(t) = e^{[V_s]t}$
A body $b$ moves with the spatial twist \( \xi_s = \left( \frac{1}{2}, 0, 0, 1, 0, 2 \right) \), what is its pose (homogeneous transformation matrix) after \( t = \frac{\pi}{2} \) secs?

\[
T_{sb} = \begin{pmatrix}
1 & 0 & 0 & \frac{\pi}{2} \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & \pi \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
T_{sb} = \begin{pmatrix}
1 & 0 & 0 & \frac{\pi}{2} \\
0 & 0 & -1 & -2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
T_{sb} = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
Transforming Clouds of Points

\[ PC_s(t_0) = T_{sb}(t_0) PC_b \]

\[ \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \]

\[ \begin{bmatrix} \frac{1}{s} & \frac{1}{b} \\ \vdots & \vdots \\ \frac{1}{s} & \frac{1}{b} \end{bmatrix} \]

\[ PC_s(t_1) = T_{sb}(t_1) PC_b \]

Stanford University
Estimating a Transformation from two Clouds of Points

1. Compute centroids $C_s, C_b$
2. Generate $H$
   \[
   H = \sum_{i}^{N} (p_s^i - C_s)(p_b^i - C_b)^T
   \]
3. Compute $R$
   \[
   [U, S, V] = \text{SVD}(H)
   \]
   \[
   R = VU^T
   \]
4. Compute $t$
   \[
   t = -R \cdot C_s + C_b
   \]
RANSAC [Fischler & Bolles, 81]

- RANdom SAmple Consensus
- Algorithm to estimate the parameters of model from data with outliers (for example, parameters of a line or of a rigid pose)
- RANSAC loop:
  - Randomly select a “seed group” of points on which to base transformation estimate (e.g., a group of matches)
  - Compute parameters from seed group
  - Find inliers to this model
  - If the number of inliers is sufficiently large, re-compute least-squares estimate of model on all of the inliers
  - Keep model with the largest number of inliers
RANSAC [Fischler & Bolles, 81]

- Approach: we want to avoid the impact of outliers, so let’s look for “inliers”, and use only those

- Intuition: if an outlier is chosen to compute the current fit, then the resulting group won’t have much support from the rest of elements
Pros and Cons of RANSAC

● Pros:
  ○ General method suited for a wide range of model fitting problems
  ○ Easy to implement and easy to calculate its failure rate

● Cons:
  ○ Only handles a moderate percentage of outliers without cost blowing up
  ○ Many real problems have a high rate of outliers (but sometimes selective choice of random subsets can help)
RANSAC Exercise - Prior

Given two points \((x_1, y_1), (x_2, y_2)\), the line connecting them is:

\[
\begin{align*}
    k &= \frac{y_2 - y_1}{x_2 - x_1} \\
    c &= -kx_1 + y_1 \\
    y &= kx + c
\end{align*}
\]

And the distance of a point \((x_0, y_0)\) to the line can be calculated as:

\[
d = \frac{|c + kx_0 - y_0|}{\sqrt{1 + k^2}}
\]
Given the points \( \{(2, 7), (0, 3), (1, 6), (1, 5)\} \), which ones form the largest set on a 2D line?

\[
\{(2, 7), (0, 3), (1, 5)\} \\
\{(2, 7), (1, 5)\} \\
\{(2, 7), (0, 3), (1, 6)\}
\]
Questions?