Recursive State Estimation

Lecture 7
Perception as a Continuous Process
Perception as a Multi-Modal Experience
Perception as Inference

Plato (I BC)

Helmholtz (1821-1894)
Recursive State Estimation

Mathematical Formalism to:

- continuously integrate measurements
- from different sensor sources
- to infer the state of a latent variable
Why is this Useful for Robotics?
Today

- Intro: why state estimation?
- Bayes Filter
- Kalman Filter
The Agent and the Environment

- Perception
- World Model
- Decision Making

Stanford University
Notation

- **State vector**: \( x \in \mathbb{R}^n \)
- **Observation**: \( z \in \mathbb{R}^k \)
- **Random Variable**: \( u \in \mathbb{R}^m \)

\[
X_t, z_{1:t}, u_{1:t} \rightarrow p(X_t | z_{1:t}, u_{1:t}) \rightarrow p(X_t | X_{t-1}, z_t, u_t)
\]
Probability Theory Refresh

\[ X, x \rightarrow p(X = x) = p(x) \]

\[ p(x) \geq 0 \]

Gaussians \[ \rightarrow p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}} \]

\[ \int p(x) dx = 1 \]

\[ \sum p(x) = 1 \]
Joint distribution

If $X$ and $Y$ are independent, then $p(x, y) = p(x)p(y)$

Conditional probability:

If $X$ and $Y$ are independent, $p(x \mid y) = p(x)$

Theorem of total probability:

\[
p(x) = \sum_y p(x \mid y) \cdot p(y) \]

\[
p(x) = \int p(x \mid y) p(y) dy
\]
Bayes' Rule:

\[
p(x|y) = \frac{p(y|x) \cdot p(x)}{\int p(y|x') \cdot p(x') dx'}
\]

\[
p(x|y) = \frac{p(y|x) \cdot p(x)}{p(y)}
\]

\[
E[X] = \sum_x x \cdot p(x)
\]

\[
E[X^2] = \int x^2 \cdot p(x) \, dx
\]

\[
Cov[X] = E[X^2] - E[X]^2
\]
The Bayes Filter

\[
\begin{align*}
\pi_t: \quad & p(x_t | z_{1:t}, u_{1:t}, x_0) = \frac{p(z_t | x_t, z_{1:t-1}, u_{1:t}, x_0) p(x_t | z_{1:t-1}, u_{1:t}, x_0)}{p(z_t | z_{1:t-1}, u_{1:t}, x_0)} \\
& = \mathcal{N} \cdot p(z_t | x_t, z_{1:t-1}, u_{1:t}, x_0) p(x_t | z_{1:t-1}, u_{1:t}, x_0) \\
& = \mathcal{N} \cdot p(z_t | x_t)
\end{align*}
\]

State is COMPLETE
We introduce $x_{t-1}$ as an additional variable.

$$p(x_t | z_{1:t-1}, u_{1:t}, x_0) = \sum_{x_{t-1}} p(x_t | x_{t-1}, z_{1:t-1}, u_{1:t}, x_0) p(x_{t-1} | z_{1:t-1}, u_{1:t}, x_0)$$

$$p(x_t | x_{t-1}, z_{1:t-1}, u_{1:t}, x_0) = p(x_t | x_{t-1}, u_t)$$

Markov

Forward model

Transition model
\[ p(x_t | z_{1:t}, u_{1:t}, x_0) = \mathbb{E}_{x_{t-1}} \left[ p(x_t | x_{t-1}, u_t) \right] \mathbf{M} \mathbb{E}_{x_{t-1}} \left[ p(x_{t-1} | z_{1:t-1}, u_{1:t-1}, x_0) \right] dx_{t-1} \]

\( \text{Bel}(x_t) \)

1. **Predict**

2. **Correct**
Limitations

1. \( p(x) \) is defined \( \forall x \in X \)
   - Defining and operating prob. dist. is usually intractable
     - Discrete and small space: OK
     - Continuous and/or large: No solutions

2. Integral term not solvable
   - Moments
     - Finite set of samples
   - No solutions
What is the predicted state?

\[ X = \{N, S, E, W\} \]
\[ p(x_0) = (0.1, 0.1, 0.1, 0.7) \]
\[ p(x_t = k|x_{t-1} = k, u_t) = 0.9 \]
\[ p(x_t = k'|x_{t-1} = k, u_t) = 0.1 \]
\[ p(z = k|x = k) = 0.9 \]
\[ p(z = k'|x = k) = 0.1 \]
\[ z_1 = N \]

\[ \hat{Bel}(x_1) = (0.1, 0.9) \]
\[ \hat{Bel}(x_1) = (0.1, 0.1, 0.1, 0.7) \]
\[ \hat{Bel}(x_1) = (0.25, 0.25, 0.25, 0.25) \]

Depends on \( u_1 \)
What is the value of $\eta$?

\[ X = \{N, S, E, W\} \]
\[ p(x_0) = (0.1, 0.1, 0.1, 0.7) \]
\[ p(x_t = k | x_{t-1} = k, u_t) = 0.9 \]
\[ p(x_t = k' | x_{t-1} = k, u_t) = 0.1 \]
\[ p(z = k | x = k) = 0.9 \]
\[ p(z = k' | x = k) = 0.1 \]

\[ z_1 = N \]

1/1.8

\[
\begin{align*}
1.8 & \\
1 & \\
1/0.18 & 
\end{align*}
\]
What is the new belief \( p(x_1 | z_1, u_1, x_0) \)?

\[
\begin{align*}
X &= \{N, S, E, W\} \\
p(x_0) &= (0.1, 0.1, 0.1, 0.7) \\
p(x_t = k | x_{t-1} = k, u_t) &= 0.9 \\
p(x_t = k' | x_{t-1} = k, u_t) &= 0.1 \\
p(z = k | x = k) &= 0.9 \\
p(z = k' | x = k) &= 0.1 \\
z_1 &= N \\
\end{align*}
\]

\[
\begin{align*}
(0.25, 0.25, 0.25, 0.25) \\
(1/18, 1/18, 1/18, 1/18) \\
(7/18, 1/18, 1/18, 9/18) \\
(9/18, 1/18, 1/18, 7/18)
\end{align*}
\]
Kalman Filter

Assumptions

- All RVs are Gaussian

\[ \Sigma \]

\[ \Rightarrow \text{Represented } \mu, \sigma^2 : N(\mu, \sigma) \Rightarrow \text{Parametric repr.} \]

\[ \Rightarrow \text{Integral can be solved analytically!} \]

- To "maintain" the Gaussians \( \Rightarrow \) Forward and measurement models \( \text{LINEAR} \)

\[ \Rightarrow \text{Solution is optimal (if assumptions are true)} \]